MATH 2050C Mathematical Analysis I

2019-20 Term 2

Midterm solution

Q1. (a) *(5 points)* Using only the field axioms of $\mathbb{R}$ (Bartle 2.1.1), prove that

$$(a+b)(a+b) = a^2 + 2(ab) + b^2$$

for any $a, b \in \mathbb{R}$. List which property you have used in each step of your argument.

Solution: We use the notations as in Bartle 2.1.1.

$$\begin{aligned}
(a+b)(a+b) &= a(a+b) + b(a+b) && (D)\\
&= (a^2+ab) + (ba+b^2) && (D)\\
&= ((a^2+ab)+ba) + b^2 && (A2)\\
&= (a^2+(ab+ba)) + b^2 && (A2)\\
&= (a^2+(ab+ab)) + b^2 && (M1)\\
&= a^2 + (ab+ab) + b^2 && (A2)\\
&= a^2 + 1(ab) + 1(ab) + b^2 && (M3)\\
&= a^2 + (1+1)(ab) + b^2 && (D)\\
&= a^2 + 2(ab) + b^2
\end{aligned}$$

(b) *(5 points)* Prove by mathematical induction that for any $n \in \mathbb{N}$, and any positive real numbers $x_1, x_2, \cdots, x_n$, we have

$$(1+x_1)(1+x_2)\cdots(1+x_n) \geq 1 + x_1 + x_2 + \cdots + x_n.$$

Solution: For $n=1$, it is trivially true as $1+x_1 = 1+x_1$. Suppose the statement is true for $n=k$. Consider $n=k+1$, we have from the induction hypothesis that

$$(1+x_1)(1+x_2)\cdots(1+x_k) \geq 1 + x_1 + x_2 + \cdots + x_k.$$

Multiplying by $1+x_{k+1}$, which is positive as $x_{k+1} > 0$, on both sides, we obtain

$$\begin{aligned}
(1+x_1)(1+x_2)\cdots(1+x_{k+1}) &\geq (1+x_1+x_2+\cdots+x_k)(1+x_{k+1})\\
&= 1 + x_1 + x_2 + \cdots + x_{k+1} + x_{k+1}(x_1+x_2+\cdots+x_k)\\
&> 1 + x_1 + x_2 + \cdots + x_{k+1}
\end{aligned}$$

where the last inequality holds because $x_1, x_2, \cdots, x_{k+1} > 0$. The proof is thus completed by mathematical induction.

Q2. *(10 points)* Use the $\epsilon - K$ definition of limit to show that

$$\lim \left(\frac{\sqrt{n}-1}{\sqrt{n}+1} \right) = 1.$$

Solution: Let $\epsilon > 0$. By Archimedean Property, we can choose $K \in \mathbb{N}$ such that $K > \frac{4}{\epsilon^2}$. Then for any $n \geq K$, we have

$$\left| \frac{\sqrt{n}-1}{\sqrt{n}+1} - 1 \right| = \left| \frac{-2}{\sqrt{n}+1} \right| \leq \frac{2}{\sqrt{n}} \leq \frac{2}{\sqrt{K}} < \epsilon.$$

Q3. (a) *(5 points)* Show that the sequence (x_n) of real numbers defined by

$$x_n := \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2^n}\right)$$

is convergent.

Solution: First, notice that for any $n \in \mathbb{N}$, we have $x_n > 0$ and

$$x_{n+1} = \left(1 - \frac{1}{2^{n+1}}\right) x_n < x_n.$$

Therefore, (x_n) is a decreasing sequence which is bounded from below by 0. By Monotone Convergence Theorem , (x_n) is a convergent sequence.

(b) *(5 points)* Suppose (x_n) is an increasing sequence of real numbers such that there exists a subsequence (x_{n_k}) which converges to $a \in \mathbb{R}$. Prove that (x_n) converges to a.

Solution: Since any subsequence of an increasing sequence is also increasing, (x_{n_k}) is an increasing sequence converging to a. By Monotone Convergence Theorem, we know that $a = \sup\{x_{n_k} : k \in \mathbb{N}\}$. We first show that $\{x_n : n \in \mathbb{N}\}$ is bounded above by a. For any $k \in \mathbb{N}$, we have $n_k \geq k$ by the definition of subsequences. Since (x_n) is increasing, we have

$$x_k \leq x_{n_k} \leq a$$

where the second inequality holds since a is the supremum, hence an upper bound, of $\{x_{n_k} : k \in \mathbb{N}\}$. Therefore, we have shown that $x_k \leq a$ for all $k \in \mathbb{N}$. As (x_n) is an increasing sequence which is bounded above, it must be convergent by Monotone Convergence Theorem. Let $\lim(x_n) = b$. We want to show that $b = a$. Since (x_n) is convergent, any subsequence of (x_n) converges to the same limit b. As we know that the subsequence (x_{n_k}) converges to a, we must have $a = b$.

Q4. (a) *(5 points)* Use the $\epsilon - H$ terminology to state the definition for a sequence of real numbers (x_n) to be NOT Cauchy.

Solution: A sequence (x_n) is NOT Cauchy if there exists some $\epsilon_0 > 0$ such that for any $H \in \mathbb{N}$, there exist some $m, n \geq H$ such that $|x_m - x_n| \geq \epsilon_0$.

(b) *(5 points)* Show that the sequence (x_n) of real numbers defined by

$$x_n := \sin 1 + \frac{\sin 2}{2!} + \frac{\sin 3}{3!} + \cdots + \frac{\sin n}{n!}$$

is convergent.

Solution: We shall show that (x_n) is a Cauchy sequence, which must then be convergent by Cauchy criteria. Let $\epsilon > 0$. Choose $H \in \mathbb{N}$ such that $H > 1 + |\log_2 \epsilon|$. Then for any $m, n \geq H$, say $m > n$, we have

$$\begin{aligned} |x_m - x_n| &= \left| \tfrac{\sin(n+1)}{(n+1)!} + \tfrac{\sin(n+2)}{(n+2)!} + \cdots + \tfrac{\sin m}{m!} \right| \\ &\leq \tfrac{|\sin(n+1)|}{(n+1)!} + \tfrac{|\sin(n+2)|}{(n+2)!} + \cdots + \tfrac{|\sin m|}{m!} \\ &\leq \tfrac{1}{(n+1)!} + \tfrac{1}{(n+2)!} + \cdots + \tfrac{1}{m!} \end{aligned}$$

where we have used Triangle inequality and the fact that $|\sin x| \leq 1$ for any $x \in \mathbb{R}$. Since $2^{r-1} \leq r!$ for all $r \in \mathbb{R}$, we have

$$|x_m - x_n| \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}} \leq \frac{1}{2^{H-1}} < \epsilon.$$

Q5. *(10 points)* Use limit theorems to find the limit

$$\lim_{x \to 1} \frac{x^3 - 3x + 2}{x^4 - 4x + 3}$$

and then use the $\epsilon - \delta$-definition of limit to justify your answer.

Solution: First, notice that

$$\frac{x^3 - 3x + 2}{x^4 - 4x + 3} = \frac{(x-1)^2(x+2)}{(x-1)^2(x^2+2x+3)} = \frac{x+2}{x^2+2x+3}.$$

As $\lim_{x \to 1}(x+2) = 3$ and $\lim_{x \to 1}(x^2+2x+3) = 6 \neq 0$, using limit theorems, we conclude that

$$\lim_{x \to 1} \frac{x^3 - 3x + 2}{x^4 - 4x + 3} = \frac{\lim_{x \to 1}(x+2)}{\lim_{x \to 1}(x^2+2x+3)} = \frac{3}{6} = \frac{1}{2}.$$

Next, we prove $\lim_{x \to 1} \frac{x^3-3x+2}{x^4-4x+3} = \frac{1}{2}$ using the $\epsilon - \delta$ definition of limit. Let $\epsilon > 0$. First notice that whenever $0 < |x-1| < 1$, we have

$$|x+1| = x+1 < 3 \quad \text{and} \quad |x^2+2x+3| > 3.$$

Therefore, if we choose $\delta = \min\{2\epsilon, 1\} > 0$, then whenever $0 < |x-1| < \delta$, we have

$$\left| \frac{x^3 - 3x + 2}{x^4 - 4x + 3} - \frac{1}{2} \right| = \left| \frac{-x^2+1}{2(x^2+2x+3)} \right| = \frac{|x+1|}{2|x^2+2x+3|}|x-1| < \frac{1}{2}\delta \leq \epsilon.$$

Q6. *(10 points)* Let (x_n) be a bounded sequence of real numbers. Define a new sequence (y_n) where

$$y_n := \inf\{x_k \ : \ k \geq n\}.$$

Define $y := \sup\{y_n \ : \ n \in \mathbb{N}\}$. Prove that there exists a subsequence (x_{n_k}) of (x_n) such that $\lim(x_{n_k}) = y$.

Solution: First, (y_n) is a bounded sequence since (x_n) is bounded. Moreover, for any $n \in \mathbb{N}$,

$$y_n := \inf\{x_k \ : \ k \geq n\} \leq \inf\{x_k \ : \ k \geq n+1\} =: y_{n+1}.$$

So (y_n) is a bounded increasing sequence. By Monotone Convergence Theorem, (y_n) converges to y.

We now construct the subsequence (x_{n_k}) inductively. By the definition of infimum, there exists some $n_1 \geq 1$ such that

$$y_1 \leq x_{n_1} < y_1 + 1.$$

After n_1 is chosen, we choose $n_2 \geq n_1 + 1$ such that

$$y_{n_1+1} \leq x_{n_2} < y_{n_1+1} + \frac{1}{2}.$$

Proceeding inductively, we then have $n_1 < n_2 < \cdots$ and a subsequence (x_{n_k}) of (x_n) such that

$$y_{n_k+1} \leq x_{n_{k+1}} < y_{n_k+1} + \frac{1}{k+1}$$

for all $k \in \mathbb{N}$. Take $k \to \infty$ and by Squeeze Theorem , we have $\lim(x_{n_k}) = \lim(y_n) = y$.

Q7. *(10 points)* Let (x_n) be a sequence of positive real numbers satisfying the following:

$$x_{m+n} \leq x_m + x_n \qquad \text{for any } m, n \in \mathbb{N}.$$

Prove that the sequence $(\frac{x_n}{n})$ is convergent.

Solution: We shall show that $\lim(\frac{x_n}{n}) = \ell$ where

$$\ell := \inf\left\{\frac{x_n}{n} : n \in \mathbb{N}\right\} \geq 0.$$

Let $\epsilon > 0$. By the definition of infimum, there exists some $q \in \mathbb{N}$ such that

$$\frac{x_q}{q} < \ell + \frac{\epsilon}{2}.$$

By division algorithm, for any $n \in \mathbb{N}$, there exists integers $p \geq 0$ and $0 \leq r < q$ such that $n = pq + r$. Set $x_0 := 0$. By assumption, we have

$$x_n = x_{pq+r} \leq x_{pq} + x_r \leq px_q + x_r.$$

Dividing by n, we obtain

$$\ell \leq \frac{x_n}{n} \leq p\frac{x_q}{n} + \frac{x_r}{n} = \frac{pq}{n}\frac{x_q}{q} + \frac{x_r}{n}.$$

Obverse that

$$\frac{pq}{n} = \frac{n-r}{n} = 1 - \frac{r}{n} \leq 1.$$

Let $M := \max\{x_1, x_2, \cdots, x_{q-1}\}$. If we choose $K \in \mathbb{N}$ such that $K > \frac{2M}{\epsilon}$, then for all $n \geq K$, we have

$$\ell \leq \frac{x_n}{n} \leq \frac{x_q}{q} + \frac{M}{n} < \left(\ell + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2} = \ell + \epsilon.$$

—END OF MIDTERM—